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LETTER TO THE EDITOR

The Virasoro and Kac–Moody symmetries for the principal chiral model

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Abstract. For the principal chiral model, two hierarchies of symmetries and the related Virasoro and Kac-Moody algebras are derived by using a recursion operator.

Recently it has been shown that some physically interesting models, such as the principal chiral model and the 4D self-dual Yang-Mills fields, possess infinitely many transformations of the hidden symmetries. The related algebra of the transformations is known to be the Kac-Moody type without central extension (see, e.g., Dolan 1981, Wu 1983, Chau 1984 and references therein). However, further investigations show that for some other models, like the non-linear σ model, there is a set of additional hidden symmetries, which is related to the Virasoro algebra (Hou and Li 1987a, b). The usual approach for all of these studies is based on the well known infinitesimal Riemann-Hilbert transform.

The interesting point is that similar algebraic structures of symmetries for many 'soliton' equations (e.g., the Korteweg-de Vries equation, etc) have also been found (see, e.g., Li and Zhu 1986, Oevel 1987, Chen and Lin 1987); however, the method used there is the explicit construction of the infinitesimal hidden symmetry transformations by means of the so-called recursion operator for the basic fields under consideration.

In the present letter, we would also like to use a recursion operator to construct hidden symmetries and prove their Kac-Moody and, especially, the Virasoro algebraic structure for the principal chiral model. In such a way, a similar result can also be obtained for the 4D self-dual Yang-Mills fields, which will be reported elsewhere.

In accordance with Bruschi *et al* (1982), let us consider the non-linear system

$$A_t = K(A) \equiv -\tfrac{1}{2}[A, \partial_x^{-1} A_t] \tag{1}$$

where $\partial_x^{-1}\partial_x = \partial_x\partial_x^{-1} = I$ and we assume here and after that $\partial_x^{-1}0 = 0$, $A \equiv A(x, t)$ is in the $N \times N$ matrix group G, $[\ ,\ ]$ being understood as the usual matrix commutator.

If we define a new field $B(x, t)$ as

$$B = -\partial_x^{-1} A_t \tag{2}$$

then (1) splits into two equations

$$A_t - B_x = [B, A] \tag{3a}$$

$$A_t + B_x = 0 \tag{3b}$$

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whereupon by letting $A = g^{-1}g_x$, $B = g^{-1}g_t$, $g \in G$, equation (3*a*) is identically satisfied and (3*b*) gives the usual form of the principal chiral model:

$$(g^{-1}g_x)_t + (g^{-1}g_t)_x = 0. \tag{4}$$

Therefore we will focus our discussion on (1). A hidden symmetry, say τ, corresponds to an infinitesimal parameter; namely, τ satisfies

$$\tau_t = K'[\tau] \tag{5}$$

where

$$K'[\tau] = \lim_{\varepsilon\to 0} \frac{\partial}{\partial\varepsilon} K(A + \varepsilon\tau)$$

is the Gateaux derivative.

In order to construct symmetries and the related Lie algebra, let us first introduce the following operator (Bruschi *et al* 1982):

$$(\mathscr{L}f)(x, t) = -\tfrac{1}{2}[A, \partial_x^{-1} f] \qquad f \in G \tag{6}$$

thus (1) can be written as $A_t = K = \mathscr{L}A_t$ and it is apparent that if A satisfies this equation, then $\mathscr{L}^n A_t = A_t$. Moreover, the operator is both a strong symmetry of (1)

$$\mathscr{L}'[K] = [K', \mathscr{L}] \tag{7}$$

and a hereditary (or Nijenhuis) operator

$$(\mathscr{L}'[\mathscr{L}f]g - \mathscr{L}\mathscr{L}'[f]g) - (\mathscr{L}'[\mathscr{L}g]f - \mathscr{L}\mathscr{L}'[g]f) = 0 \tag{8}$$

for any $f, g \in G$. These definitions can be found in Magri (1980) and Fuchssteiner and Fokas (1981), and the proofs of (7) and (8) are as follows. Using

$$K'[f] = -\tfrac{1}{2}([f, \partial_x^{-1} A_t] + [A, \partial_x^{-1} f_t]) \tag{9}$$

and

$$\mathscr{L}'[f]g = -\tfrac{1}{2}[f, \partial_x^{-1} g] \tag{10}$$

for any $f, g \in G$, we find

$$\mathscr{L}'[K]f = \tfrac{1}{4}[[A, \partial_x^{-1} A_t], \partial_x^{-1} f] \tag{11}$$

$$K'[\mathscr{L}f] = \tfrac{1}{4}([[A, \partial_x^{-1} f], \partial_x^{-1} A_t] + [A, \partial_x^{-1}[A_t, \partial_x^{-1} f] + [A, \partial_x^{-1}[A, \partial_x^{-1} f_t]]) \tag{12}$$

$$\mathscr{L}K'[f] = \tfrac{1}{4}([A, \partial_x^{-1}[f, \partial_x^{-1} A_t]] + [A, \partial_x^{-1}[A, \partial_x^{-1} f_t]]) \tag{13}$$

and so, by the Jacobi identity, $(\mathscr{L}'[K] - [K', \mathscr{L}])f = 0$ for any $f \in G$ this implies (7). For (8), we first have

$$\mathscr{L}'[\mathscr{L}f]g = \tfrac{1}{4}([[A, \partial_x^{-1} f], \partial_x^{-1} g]) \tag{14}$$

$$\mathscr{L}\mathscr{L}'[f]g = \tfrac{1}{4}([A, \partial_x^{-1}[f, \partial_x^{-1} g]]) \tag{15}$$

so

$$\mathscr{L}'[\mathscr{L}f]g - \mathscr{L}\mathscr{L}'[f]g = \tfrac{1}{4}([[A, \partial_x^{-1} f], \partial_x^{-1} g] - [A, \partial_x^{-1}[f, \partial_x^{-1} g]]) \tag{16}$$

and similarly, we have

$$\mathscr{L}'[\mathscr{L}g]f - \mathscr{L}\mathscr{L}'[g]f = \tfrac{1}{4}([[A, \partial_x^{-1} g], \partial_x^{-1} f] - [A, \partial_x^{-1}[g, \partial_x^{-1} f]]). \tag{17}$$

Therefore, the left-hand side of (8) is

$$\tfrac{1}{4}([A, [\partial_x^{-1}f, \partial_x^{-1}g] - \partial_x^{-1}([f, \partial_x^{-1}g] - [g, \partial_x^{-1}f])])$$
$$= \tfrac{1}{4}([A, \partial_x^{-1}([\partial_x^{-1}f, \partial_x^{-1}g]_x - [f, \partial_x^{-1}g] - [g, \partial_x^{-1}f])])$$
$$= 0.$$

Equation (7) implies that the operator $\mathscr{L}$ maps the symmetry of (1) to another one, i.e. if τ satisfies (4), then so does $\mathscr{L}\tau$:

$$(\mathscr{L}\tau)_t = \mathscr{L}_t\tau + \mathscr{L}\tau_t = \mathscr{L}'[K]\tau + \mathscr{L}K'[\tau]$$
$$= K'[\mathscr{L}\tau].$$

Therefore, $\mathscr{L}$ can generate a hierarchy of symmetry if the first one is known.

It can easily be checked that

$$\tau_a^{(0)} = [A, \sigma_a] \tag{18}$$
$$\tau^{(0)} = \mathscr{L}A - A \tag{19}$$

are symmetries, where σ_a are generators of the Lie algebra of G, satisfying $[\sigma_a, \sigma_b] = C_{ab}^c\sigma_c$. Thus two hierarchies of symmetries are as follows:

$$\tau_a^{(n)} = \mathscr{L}^n\tau_a^{(0)} \tag{20}$$
$$\tau^{(n)} = \mathscr{L}^n\tau^{(0)} \tag{21}$$

for $n = 0, 1, 2, \ldots$.

In order to discuss the algebraic structures of these symmetries, let us first prove

$$[\![\tau_a^{(0)}, \bar{\tau}^{(0)}]\!] = 0 \tag{22}$$

and

$$\mathscr{L}'[\tau_a^{(n)}] = [(\tau_a^{(n)})', \mathscr{L}] \tag{23}$$
$$\mathscr{L}'[\bar{\tau}^{(n)}] = [(\bar{\tau}^{(n)})', \mathscr{L}] + \mathscr{L}^{n+1} \tag{24}$$

where $\bar{\tau}^{(n)} = \mathscr{L}^nA$, $n = 0, 1, 2, \ldots$, and the Lie bracket $[\![\ ,\ ]\!]$ is defined to be $[\![F, G]\!] = F'[G] - G'[F]$. Equation (22) can easily be checked. For $n = 0$, equation (24) can be checked directly, and if it is valid for any integer n, then for $n+1$ and any $f \in G$:

$$\mathscr{L}'[\bar{\tau}^{(n+1)}]f = \mathscr{L}'[\mathscr{L}\bar{\tau}^{(n)}]f$$
$$(\bar{\tau}^{(n+1)})'[\mathscr{L}f] = \mathscr{L}'[\mathscr{L}f]\bar{\tau}^{(n)} + \mathscr{L}(\bar{\tau}^{(n)})'[\mathscr{L}f]$$
$$\mathscr{L}(\bar{\tau}^{(n+1)})'f = \mathscr{L}\mathscr{L}'[f]\bar{\tau}^{(n)} + \mathscr{L}\mathscr{L}(\bar{\tau}^{(n)})'f.$$

Therefore, by using the hereditary property of $\mathscr{L}$, we have

$$(\mathscr{L}'[\bar{\tau}^{(n+1)}] - [(\bar{\tau}^{(n+1)})', \mathscr{L}])f$$
$$= \mathscr{L}'[\mathscr{L}\bar{\tau}^{(n)}]f - \mathscr{L}'[\mathscr{L}f]\bar{\tau}^{(n)} + \mathscr{L}\mathscr{L}'[f]\bar{\tau}^{(n)} - \mathscr{L}[(\bar{\tau}^{(n)})', \mathscr{L}]f$$
$$= \mathscr{L}(\mathscr{L}'[\bar{\tau}^{(n)}] - [(\bar{\tau}^{(n)})', \mathscr{L}]f = \mathscr{L}^{n+2}f.$$

This completes our proof. Similarly, we can verify (23).

Also by using the hereditary property of $\mathscr{L}$, and (22)-(24), we can obtain

$$[\![\tau_a^{(m)}, \tau_b^{(n)}]\!] = C_{ab}^c\tau_c^{(m+n)} \tag{25}$$
$$[\![\tau_a^{(m)}, \bar{\tau}^{(n)}]\!] = m\tau_a^{(m+n)} \tag{26}$$
$$[\![\bar{\tau}^{(m)}, \bar{\tau}^{(n)}]\!] = (m-n)\bar{\tau}^{(m+n)}. \tag{27}$$

Since $\tau^{(n)} = \bar{\tau}^{(n+1)} - \bar{\tau}^{(n)}$, it immediately implies

$$[\![\tau_a^{(m)}, \tau_b^{(n)}]\!] = C^c_{ab}\tau_c^{(m+n)} \tag{28}$$

$$[\![\tau_a^{(m)}, \tau^{(n)}]\!] = m(\tau_a^{(m+n+1)} - \tau_a^{(m+n)}) \tag{29}$$

$$[\![\tau^{(m)}, \tau^{(n)}]\!] = (m-n)(\tau^{(m+n+1)} - \tau^{(m+n)}). \tag{30}$$

Obviously, the infinite numbers of quantities $\tau_a^{(n)}$ and $\bar{\tau}^{(n)}$ form the Kac–Moody and Virasoro algebras, which lead to the Kac–Moody and an analogous Virasoro algebra for symmetries of (1).

In summary, we would like to point out that the first set of symmetries of (20) and their algebraic structure (28) have been obtained by the infinitesimal Riemann–Hilbert transform, but the second ones in (21) and the commutators of (29) and (30) are really new (to the author's knowledge). There should exist an extended infinitesimal Riemann–Hilbert transform to derive (21), (29) and (30)†; however, the approach based on the hereditary operator seems to be more straightforward.

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† See Zhang (1988) where, in terms of the infinitesimal Riemann–Hilbert transform, more complicated expressions of symmetries and their Lie algebraic structures have been obtained by lengthy calculations.